Selected Solutions for Chapter 16: Greedy Algorithms

Solution to Exercise 16.1-4

Let S be the set of n activities.

The "obvious" solution of using GREEDY-ACTIVITY-SELECTOR to find a maximum-size set S_1 of compatible activities from S for the first lecture hall, then using it again to find a maximum-size set S_2 of compatible activities from $S - S_1$ for the second hall, (and so on until all the activities are assigned), requires $\Theta(n^2)$ time in the worst case. Moreover, it can produce a result that uses more lecture halls than necessary. Consider activities with the intervals $\{[1, 4), [2, 5), [6, 7), [4, 8)\}$. GREEDY-ACTIVITY-SELECTOR would choose the activities with intervals $[1, 4)$ and $[6, 7)$ for the first lecture hall, and then each of the activities with intervals $[2, 5)$ and $[4, 8)$ would have to go into its own hall, for a total of three halls used. An optimal solution would put the activities with intervals $[1, 4)$ and $[4, 8)$ into one hall and the activities with intervals $[2, 5)$ and $[6, 7)$ into another hall, for only two halls used.

There is a correct algorithm, however, whose asymptotic time is just the time needed to sort the activities by time—$O(n \lg n)$ time for arbitrary times, or possibly as fast as $O(n)$ if the times are small integers.

The general idea is to go through the activities in order of start time, assigning each to any hall that is available at that time. To do this, move through the set of events consisting of activities starting and activities finishing, in order of event time. Maintain two lists of lecture halls: Halls that are busy at the current event-time t (because they have been assigned an activity i that started at $s_i \le t$ but won't finish until $f_i > t$) and halls that are free at time t. (As in the activity-selection problem in Section 16.1, we are assuming that activity time intervals are half open—i.e., that if $s_i \ge f_j$, then activities i and j are compatible.) When t is the start time of some activity, assign that activity to a free hall and move the hall from the free list to the busy list. When t is the finish time of some activity, move the activity's hall from the busy list to the free list. (The activity is certainly in some hall, because the event times are processed in order and the activity must have started before its finish time t, hence must have been assigned to a hall.)

To avoid using more halls than necessary, always pick a hall that has already had an activity assigned to it, if possible, before picking a never-used hall. (This can be done by always working at the front of the free-halls list—putting freed halls onto

the front of the list and taking halls from the front of the list—so that a new hall doesn't come to the front and get chosen if there are previously-used halls.)

This guarantees that the algorithm uses as few lecture halls as possible: The algorithm will terminate with a schedule requiring $m \leq n$ lecture halls. Let activity i be the first activity scheduled in lecture hall m. The reason that i was put in the mth lecture hall is that the first $m-1$ lecture halls were busy at time s_i. So at this time there are m activities occurring simultaneously. Therefore any schedule must use at least m lecture halls, so the schedule returned by the algorithm is optimal.

Run time:

- Sort the $2n$ activity-starts/activity-ends events. (In the sorted order, an activity-ending event should precede an activity-starting event that is at the same time.) $O(n \lg n)$ time for arbitrary times, possibly $O(n)$ if the times are restricted (e.g., to small integers).
- Process the events in $O(n)$ time: Scan the $2n$ events, doing $O(1)$ work for each (moving a hall from one list to the other and possibly associating an activity with it).

Total: $O(n + \text{time to sort})$

Solution to Exercise 16.2-2

The solution is based on the optimal-substructure observation in the text: Let i be the highest-numbered item in an optimal solution S for W pounds and items $1, \ldots, n$. Then $S' = S - \{i\}$ must be an optimal solution for $W - w_i$ pounds and items $1, \ldots, i-1$, and the value of the solution S is v_i plus the value of the subproblem solution S'.

We can express this relationship in the following formula: Define $c[i, w]$ to be the value of the solution for items $1, \ldots, i$ and maximum weight w. Then

$$c[i,w] = \begin{cases} 0 & \text{if } i = 0 \text{ or } w = 0\,, \\ c[i-1,w] & \text{if } w_i > w\,, \\ \max(v_i + c[i-1, w-w_i], c[i-1,w]) & \text{if } i > 0 \text{ and } w \geq w_i\,. \end{cases}$$

The last case says that the value of a solution for i items either includes item i, in which case it is v_i plus a subproblem solution for $i-1$ items and the weight excluding w_i, or doesn't include item i, in which case it is a subproblem solution for $i-1$ items and the same weight. That is, if the thief picks item i, he takes v_i value, and he can choose from items $1, \ldots, i-1$ up to the weight limit $w - w_i$, and get $c[i-1, w-w_i]$ additional value. On the other hand, if he decides not to take item i, he can choose from items $1, \ldots, i-1$ up to the weight limit w, and get $c[i-1, w]$ value. The better of these two choices should be made.

The algorithm takes as inputs the maximum weight W, the number of items n, and the two sequences $v = \langle v_1, v_2, \ldots, v_n \rangle$ and $w = \langle w_1, w_2, \ldots, w_n \rangle$. It stores the $c[i, j]$ values in a table $c[0\,..\,n, 0\,..\,W]$ whose entries are computed in row-major order. (That is, the first row of c is filled in from left to right, then the second row,

and so on.) At the end of the computation, $c[n, W]$ contains the maximum value the thief can take.

```
DYNAMIC-0-1-KNAPSACK(v, w, n, W)
let c[0..n, 0..W] be a new array
for w = 0 to W
    c[0, w] = 0
for i = 1 to n
    c[i, 0] = 0
    for w = 1 to W
        if w_i ≤ w
            if v_i + c[i − 1, w − w_i] > c[i − 1, w]
                c[i, w] = v_i + c[i − 1, w − w_i]
            else c[i, w] = c[i − 1, w]
        else c[i, w] = c[i − 1, w]
```

We can use the c table to deduce the set of items to take by starting at $c[n, W]$ and tracing where the optimal values came from. If $c[i, w] = c[i-1, w]$, then item i is not part of the solution, and we continue tracing with $c[i-1, w]$. Otherwise item i is part of the solution, and we continue tracing with $c[i-1, w-w_i]$.

The above algorithm takes $\Theta(nW)$ time total:

- $\Theta(nW)$ to fill in the c table: $(n+1)\cdot(W+1)$ entries, each requiring $\Theta(1)$ time to compute.
- $O(n)$ time to trace the solution (since it starts in row n of the table and moves up one row at each step).

Solution to Exercise 16.2-7

Sort A and B into monotonically decreasing order.

Here's a proof that this method yields an optimal solution. Consider any indices i and j such that $i < j$, and consider the terms ${a_i}^{b_i}$ and ${a_j}^{b_j}$. We want to show that it is no worse to include these terms in the payoff than to include ${a_i}^{b_j}$ and ${a_j}^{b_i}$, i.e., that ${a_i}^{b_i}{a_j}^{b_j} \geq {a_i}^{b_j}{a_j}^{b_i}$. Since A and B are sorted into monotonically decreasing order and $i < j$, we have $a_i \geq a_j$ and $b_i \geq b_j$. Since a_i and a_j are positive and $b_i - b_j$ is nonnegative, we have ${a_i}^{b_i-b_j} \geq {a_j}^{b_i-b_j}$. Multiplying both sides by ${a_i}^{b_j}{a_j}^{b_j}$ yields ${a_i}^{b_i}{a_j}^{b_j} \geq {a_i}^{b_j}{a_j}^{b_i}$.

Since the order of multiplication doesn't matter, sorting A and B into monotonically increasing order works as well.